DIOPHANTINE ANALYSIS.

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SOLUTIONS OF PROBLEMS.

47. Proposed by M. A. GRUBER, A. M., War Department, Washington D. C.

Find the first six sets of values in which the sum of two consecutive integral squares equals a square.

I. Solution by ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

Let $\frac{1}{2}(x_n-1)$ and $\frac{1}{2}(x_n+1)$ be two consecutive integers, and y_n^2 the sum of their squares; then we must have

$$\tfrac{1}{4}(x_n-1)^2+\tfrac{1}{4}(x_n+1)^2=y_n^2, \text{ or } x_n^2-2y_n^2=-1 \quad\text{.............(1)},$$

which may be written

$$(x_n-y_n\sqrt{2})(x_n+y_n\sqrt{2})=-1 \quad\text{..................(2)}.$$

When $n=1$, we have

$$(x_1-y_1\sqrt{2})(x_1+y_1\sqrt{2})=-1 \quad\text{..................(3)};$$

also, raising (3) to the $(2n+1)$th power, we have

$$(x_1-y_1\sqrt{2})^{2n+1}(x_1+y_1\sqrt{2})^{2n+1}=-1 \quad\text{................(4)},$$

where n may be 0, 1, 2, 3, 4, etc.

Assuming $x_n-y_n\sqrt{2}=(x_1-y_1\sqrt{2})^{2n+1}$,

$x_n+y_n\sqrt{2}=(x_1+y_1\sqrt{2})^{2n+1}$, as we are at liberty to do, we find

$$x_n=[(x_1+y_1\sqrt{2})^{2n+1}+(x_1-y_1\sqrt{2})^{2n+1}]/2,$$

$$y_n=[(x_1+y_1\sqrt{2})^{2n+1}-(x_1-y_1\sqrt{2})^{2n+1}]/2\sqrt{2}.$$

It is easily seen that $x_1=1$ and $y_1=1$; therefore

$$x_n=[(\sqrt{2}+1)^{2n+1}-(\sqrt{2}-1)^{2n+1}]/2,\quad y_n=[(\sqrt{2}+1)^{2n+1}+(\sqrt{2}-1)^{2n+1}]/2\sqrt{2},$$

and the required numbers are

$$\tfrac{1}{4}[(\sqrt{2}+1)^{2n+1}-(\sqrt{2}-1)^{2n+1}-2] \text{ and } \tfrac{1}{4}[(\sqrt{2}+1)^{2n+1}-(\sqrt{2}-1)^{2n+1}+2].$$

The operation of involution is very tedious except when n is a small number. When x_n and y_n are very large numbers we have from (1) very nearly

$x_n/y_n = \sqrt{2}$, and the values of x_n and y_n are the numerators and denominators of the odd convergents to $\sqrt{2}$ expanded as a continued fraction, after the first. The successive odd convergents are

$$1/1,\ 7/5,\ 41/29,\ 239/169,\ 1393/985,\ 8119/5741, \text{ etc.}$$

The values of x_n and y_n are connected by the relations

$$x_n = 6x_{n-1} - x_{n-2} \ldots\ldots (5), \quad y_n = 6y_{n-1} - y_{n-2} \ldots\ldots (6),$$

which afford an easy method of computing the successive sets of numbers required. When $n=1$ we easily find from the general formulas the first set, 3 and 4, and then from (5) the successive sets, which are

2nd set	20,	21,
3d "	119,	120,
4th "	696,	697,
5th "	4059,	4060,
6th "	23660,	23661.

See the *Mathematical Visitor*, Vol. I., No. 3, page 56, where the fifth and sixth sets are erroneously given as 4058, 4059 and 23657, 23658. The root of the sum of the squares of the sixth set should be 33461 instead of 33457. The 100 set is given on the same page; and also on page 122 where the numbers are found by Mr. K. S. Putnam by a different method.

These numbers solve the geometrical problem—"To find rational right-angled triangles whose legs are consecutive numbers."

II. Solution by A. H. BELL, Hillsboro, Illinois.

We have $x^2+(x+1)^2=\square$, or $2x^2+2x+1=\square$ ....(1).

Take $Ax^2 \pm Bx + C = \square = y^2$ ....(2).

$(2)\times A$, and add and subtract, etc.

$(Ax \pm B/2)^2 = Ay^2 + B^2/4 - AC = \square = t^2$ ....(3).

$\therefore t^2 - Ay^2 = B^2/4 - AC$ ....(A).

Also $x = (t \mp B/2)/A$ ....(B).

Let (A) reduce to $t^2 - Ay^2 = \pm D$ ....(4);

and a complete quotient$=(\sqrt{A}+M)/D$. Then the preceding convergent will $=t/y$ and will answer the $+$ or $-D$ as it is an odd or even number of fraction.

Also $v^2 - Au^2 = 1$ ....(5).

$(4)\times(5)$, and add and subtract $2Auvyt$, etc.

$(vt \pm Ayu)^2 - A(ut \pm vy)^2 = \pm D$, or $t_n{}^2 - Ay_n{}^2 = \pm D$ ....(C).

But in the problem (A) is $t^2-2y^2=-1$ ....(6).

Then we also have $t_n/y_n=(2vt_{n-1}-t_{n-2})/(2vy_{n-1}-y_{n-2})$ ....(7).

$\sqrt{2}$. Number of complete fractions=integer : 1, 2, etc.

Complete quotients=$(\sqrt{2}+0)/1$: $(\sqrt{2}+1)/1$, $(\sqrt{2}+1)/1$, etc.

Partial quotients=1 : 2, 2, etc.

Convergents=1/0, 1/1 : 3/2, 7/5, etc.

$\therefore$ $t/y=1/1$, 7/5, etc. $v/u=1/0$, 3/2. $2v_1=6$, $v_0=1$.

$\therefore$ (C) or (7) $t=1$, 7, 41, 239, 1393, 8119, 47321, etc.

(B) $x=0/2$, 3, 20, 119, 696, 4059, 23660, etc.

$x+1$ will give the six sets of values required.

NOTE. $2v=M=$magic M of Roberts and Robins.

III. Solution by the PROPOSER.

Take the formula for finding the sum of two integral squares equal to a square :

$$(2mp)^2+(m^2-n^2)^2=(m^2+n^2)^2 \text{..................} (A).$$

Then will the difference between $2mn$ and m^2-n^2 be 1. When $m^2-n^2>2mn$, we have $m^2-n^2-2mn=1$; whence $m=n\pm\sqrt{2n^2+1}$. When $2mn>m^2-n^2$, we have $2mn-(m^2-n^2)=1$, whence $m=n\pm\sqrt{2n^2-1}$. Substituting these values of m in (A), we obtain

$$[2n(n\pm\sqrt{2n^2\pm1})]^2+[2n(n\pm\sqrt{2n^2\pm1})\pm1]^2$$
$$=[2n(n\pm\sqrt{2n^2\pm1})+2n^2\pm1]^2 \text{........} (B).$$

It now remains to make $2n^2+1=\square$, and $2n^2-1=\square$. We find, by inspection, that the first value of n in $2n^2+1=\square$, is 0, and in $2n^2-1=\square$, is 1. Knowing these two values, we find the succeeding values from the formula, $n=2n_1+n_2$, in which n_1 is the last found known value of n, and n_2 the value just preceding. Whence $n=0$, 1, 2, 5, 12, 29, 70, 169, 408, 985, etc. *Zero* and the *even* numbers are the values of n in $2n^2+1=\square$, and the *odd* numbers in $2n^2-1=0$; the first two values of each series being known, the succeeding values can be found by the formula, $n=6n_1-n_2$.

It is also noticeable that the consecutive *odd number* values of n are the consecutive values of the root of the square that equals the sum of two consecutive integral squares. Substituting, now, the values of n in (B), we obtain, respectively, $0^2+1^2=1^2$, $4^2+3^2=5^2$, $20^2+21^2=29^2$, $120^2+119^2=169^2$, $696^2+697^2=985^2$, $4060^2+4059^2=5741^2$, $23660^2+23661^2=33461^2$, etc.

Or, from solution III of Problem 36, Vol. III., No. 3, page 82, we find that when one of the triangular square numbers is taken as $n(n+1)/2$, the next in order, is terms of n is $[2n+1+3\sqrt{n(n+1)/2}]^2$. The difference of the roots of these two successive triangular square numbers is $2n+1+2\sqrt{n(n+1)/2}$. The sum of the roots is $2n+1+4\sqrt{n(n+1)/2}$, which, when $n(n+1)/2=\square$, equals the sum of the two *consecutive integral* numbers, $n+2\sqrt{n(n+1)/2}$ and $n+1+2\sqrt{n(n+1)/2}$.

But $[n+2\sqrt{n(n+1)/2}]^2+[n+1+2\sqrt{n(n+1)/2}]^2$

$$=6n^2+6n+1+(8n+4)\sqrt{n(n+1)/2}=[2n+1+2\sqrt{n(n+1)/2}]^2.$$

We here have a general formula in which the sum of the squares of two consecutive integers equals a square. To obtain integral numerical results, we assign the successive values of n in $n(n+1)/2=\square$, 1, 8, 49, 288, 1681, 9800, etc. Whence we have $3^2+4^2=5^2$; $20^2+21^2=29^2$; $119^2+120^2=169^2$; $696^2+697^2=985^2$; $4059^2+4060^2=5741^2$; $23660^2+23661^2=33461^2$, etc.

IV. Solution by Hon. J. H. DRUMMOND, LL. D., Portland, Maine.

Let x=one number and $x+1$=the other; then x^2+x^2+2x+1 will be the sum of two consecutive squares. Then $2x^2+2x+1=\square=$(say) $(mx-1)^2$, from which we readily obtain $x=2(m+1)/(m^2-2)$. It is readily seen that x is integral when $m=2$. Then we have 2/1, 10/7, 58/41, etc., for values of m which give integral values of x, viz., 3, 119, 4059, etc. The other series which makes x integral is 3/2, 17/12, 99/70, etc., and $x=20$, 696, 23660, etc. The six values of x, therefore, are 3, 20, 119, 696, 4059, 23660, and of $x+1$, 4, 21, 120, 697, 4060, 23661, and the squares of these values are *probably* the squares required. I say "probably," because it cannot be mathematically determined that some other method of solution will not give other results that show that there are other values less than 23660 besides those I have given.

[The following is my formula for obtaining integral values of a fraction whose denominator is p^2-2, which I assume in this solution. I have never seen the formula in print and do not know how generally it is known.

If r/s is of such a value of p as will give an integral result, then $(3r+4s)/(2r+3s)$ is *another* value of p that gives an integral result, and so on *ad infinitum*. If the numerator is even, there will be two different series of values of p, the initial term in one being 2/1, and in the other 3/2; if the numerator is odd, the series beginning with 3/2 will give integral results. By means of this formula an infinite number (mathematically speaking) of integral values of x may be obtained in the equation $2x^2+ax+b^2=\square$, in terms of a and b; and in the equation $2x^2+2ax+b^2$, two series (infinite) of integral values in terms of a and b may be obtained. In both cases, however, the numbers increase in value very rapidly.]

V. Solution by A. H. HOLMES, Brunswick, Maine.

$x^2+(x+1)^2=\square$ or $2x^2+2x+1=\square$. Let $x=y+p$.

$\therefore$ $2y^2+(4p+2)y+2p^2+2p+1=\square$, from which we find the law of the series to be: $b=1+3a+2\sqrt{2a^2+2a+1}$. Let $a=3$ and we find $b=20$. Then by the same law, $c=119$, $d=696$, $e=4059$, and $f=23660$. Therefore, we have for the first six sets of values: (3 and 4), (20 and 21), (119, 120), (696 and 697), (4059 and 4060), and (23660 and 23661).

VI. Solution by H. C. WILKES, Skull Run, West Virginia.

We have $x^2+(x+1)^2=y^2=4n+1$, then $x(x+1)/2=n$. Substituting this value for n in $4n+1=y^2$ we have $x^2+x=(y^2-1)/2$. Putting $x+(x+1)=t$ or $x=(t-1)/2$, we obtain $t^2-2y^2=-1$. Since $t=7$, $y=5$ satisfy this equation, the first values of $x+(x+1)$ and y will be 7 and 5.

$\therefore$ $3^2+4^2=5^2$. From inspection of solution II, Problem 36, Vol. III, page 81, we find a formula for obtaining the succeeding values of $x+(x+1)$ and y.

$x++(x1)$.	y.	
$6\times7-1=41$,	$6\times5-1=29$,	$20^2+21^2=29^2$,
$6\times41-7=239$,	$6\times29-5=169$,	$119^2+120^2=169^2$,
$6\times239-41=1393$,	$6\times169-29=985$,	$696^2+697^2=985^2$,
$6\times1393-239=8119$,	$6\times985-169=5741$,	$4059^2+4060^2=5741^2$,
$6\times8119-2392=47321$.	$6\times5741-985=33461$.	$23660^2+23661^2=33461^2$.

Also solved by *J. SCHEFFER* and *G. B. M. ZERR.*

48. Proposed by B. F. YANNEY, A. M., Professor of Mathematics in Mount Union College, Alliance, O.

If any positive integral number N be divided by another positive integral number D, leaving a remainder 1, then any positive integral power of N, divided by D, will leave a remainder of 1.

I. Solution by ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

Let $N=nD+1$, then

$$(nD+1)^m=n^mD^m+mn^{m-1}D^{m-1}+\frac{m(m-1)}{2}n^{m-2}D^{m-2}+$$

$$\ldots\ldots+\frac{m(m-1)}{2}n^2D^2+mnD+1,$$

$$=D[n^mD^{m-1}+mn^{m-1}D^{m-2}+\frac{m(m-1)}{2}n^{m-2}D^{m-3}+\ldots\ldots+\frac{m(m-1)}{2}n^2D+mn]+1,$$

which proves the proposition.

Solved in a similar manner by *M. A. GRUBER* and *G. B. M. ZERR.*